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For All Real μ, $\ddot{x} + \mu \sin \dot{x} + x = 0$ has an Infinite Number of Limit Cycles

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The fact that

$$\dot{x} = y, \qquad \dot{y} = -x - \mu \sin y \tag{1}$$

has an infinite number of limit cycles seems to have been first conjectured by Eckweiler [*1*]. However, Eckweiler was primarily interested in existence of limit cycles for equations of the form (1) with $\sin y$ replaced by an odd polynomial in y, and $|\mu|$ small; his perturbation techniques provided proofs in this case of a finite number of limit cycles, but failed to yield a proof that (1) has an infinite number of limit cycles, even for small $|\mu|$. Following Eckweiler, Stoker ([*4*], p. 133) gave a sketch of the phase portrait of (1) for the case $\mu = -1$, again with no proofs. Twenty years after Eckweiler, Hochstadt and Stephan [*2*] succeeded in proving that for $|\mu|$ sufficiently small, (1) has an infinite number of limit cycles, periodic solutions of arbitrarily large amplitude. The methods of Hochstadt and Stephan resemble those of the present paper in some ways. Writing $x = r \cos \theta, y = r \sin \theta$, they put (1) in polar coordinates and consider r as a function of θ, obtaining the first order equation

$$\frac{dr(\theta)}{d\theta} = \frac{\mu \sin \theta \sin(r \sin \theta)}{1 + \dfrac{\mu}{r} \cos \theta \sin(r \sin \theta)}$$

(which incidentally corrects an error in (9) of [*2*]) with initial value $r(0) = A$ and corresponding solution $r(\theta, A)$. Letting $0 = A_0 < A_1 < A_2 < \cdots$ denote the zeros of the Bessel function $J_1(A)$, they use a battery of comparisons and sophisticated estimates to prove that there exists $\bar{\mu} > 0$ such that for $0 < \mu < \bar{\mu}$ and A_{2n+1} sufficiently large,

$$A_{2n+1} < r\left(2\pi, A_{2n+1} + \frac{\pi}{2}\right) < r\left(0, A_{2n+1} + \frac{\pi}{2}\right) = A_{2n+1} + \frac{\pi}{2},$$

$$A_{2n+1} - \frac{\pi}{2} = r\left(0, A_{2n+1} - \frac{\pi}{2}\right) < r\left(2\pi, A_{2n+1} - \frac{\pi}{2}\right) < A_{2n+1}.$$

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It follows that the phase plane trajectories of (1) map the interval $[A_{2n+1} - \pi/2, A_{2n+1} + \pi/2]$ continuously into itself, so that by the Brouwer fixed point theorem there is a fixed point A^* in this interval such that $r(0, A^*) = r(2\pi, A^*)$ and the corresponding orbit is a limit cycle. The present author prefers to state part of the above argument in an equivalent and simpler way: the function $r(2\pi, A) - r(0, A)$ is continuous as a function of A, negative at $A = A_{2n+1} + \pi/2$, positive at $A = A_{2n+1} - \pi/2$, therefore zero at some intermediate value A^* corresponding to a periodic orbit. (The one-dimensional Brouwer fixed point theorem is the same as the intermediate value theorem.) We assume above and shall assume in the sequel that the reader is familiar with the fact that solutions of the differential equations considered here are continuous functions of the initial conditions.

The present paper, of course, proves the result stated in the title. To be more exact, we shall prove, in slightly altered notation, that $r(2\pi, A) - r(0, A)$ is an oscillating function of A, alternately positive and negative at points spaced roughly π units apart.

Proving that (1) has an infinite number of limit cycles does not finish even the geometric analysis of $\ddot{x} + \mu \sin \dot{x} + x = 0$, for everyone who has worked with the equation agrees in conjecturing that the limit cycles crossing the axes roughly every π units are unique and alternately stable and unstable. Formulas (6) and (7) below suggest that this conjecture is correct, but no proof is yet available.

Levinson and Smith [3] observe that along orbits of (1) we have

$$\frac{dy}{dx} = \frac{-x - \mu \sin y}{y}, \qquad \frac{dx}{dy} = \frac{y}{-x - \mu \sin y}, \tag{2}$$

so that defining $u(x, y) = \frac{1}{2}(x^2 + y^2)$, along an orbit

$$\frac{du}{dx} = x + y\frac{dy}{dx} = -\mu \sin y. \tag{3}$$

Because of the simplicity of (3), we will work here with $u(x, y)$ instead of the function $r(\theta)$ discussed above. Furthermore, we will avoid polar coordinates and concentrate attention on the function $g(y)$ defined below, a method believed to be new. The "bounding semicircles" to be described below appear almost trivial, but are apparently new here, and provide the key to our analysis for large $|\mu|$.

We will work only with the upper half ($y \geqslant 0$) of the phase plane. Since (1) is invariant under substituting $-x$ for x and $-y$ for y, statements about trajectories in $y \geqslant 0$ translate directly into statements about trajectories in $y \leqslant 0$.

We begin by getting a rough geometric picture of the trajectories in $y \geqslant 0$, obtaining preliminary "semi-circle bounds", and giving some definitions. The curve $x = -\mu \sin y$ is called the "characteristic". It separates the half plane $y \geqslant 0$ into two regions, $\{(x, y): -\infty < x < -\mu \sin y,\ y \geqslant 0\}$ and $\{(x, y): -\mu \sin y < x < \infty, y \geqslant 0\}$, which we will call to the "left" or "right" of the characteristic, respectively. Specifying a point $(-\mu \sin y_0, y_0)$, $y_0 > 0$, specifies a set of initial values $x(0) = -\mu \sin y_0$, $y(0) = y_0$, therefore a unique solution to (1); the directed curve through $(-\mu \sin y_0, y_0)$ which is that portion of the corresponding trajectory for which $y \geqslant 0$ we shall call "the orbit corresponding to y_0". Equations (1) and (2) show that all orbits have slope zero on the characteristic, so that no orbit is tangent to the characteristic, and no orbit moves from right to left of the characteristic with increasing time. It follows that the orbit corresponding to y_0 crosses the characteristic just once, at $(-\mu \sin y_0, y_0)$. By (2), the orbit corresponding to y_0 has positive slope left of the characteristic and negative slope right of the characteristic. See Fig. 1. Note that we ignore the stationary point (0, 0).

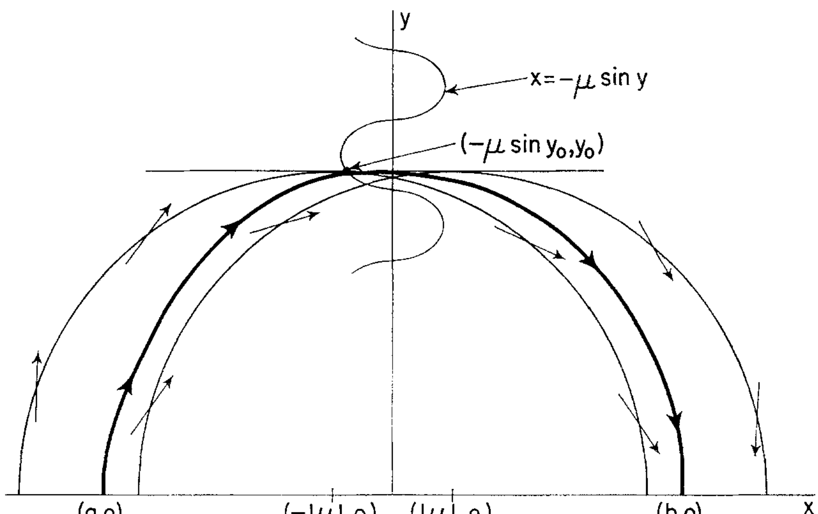

FIG. 1

Along an orbit in $y \geqslant 0$, y is therefore a single valued function of x, which we denote by $y(x)$. To the left of the characteristic, x is a single valued function of y, which we denote by $x_-(y)$. To the right of the characteristic, x is a single valued function of y, which we denote by $x_+(y)$. We define

$$g(y) = \frac{dx_-(y)}{dy} - \frac{dx_+(y)}{dy}. \tag{4}$$

Clearly $g(y) > 0$ for $y > 0$. We suppress the dependence of $y(x)$, $x_-(y)$, $x_+(y)$, and $g(y)$ on y_0.

Consider a semi-circle

$$y_c(x) = (r^2 - (x - c)^2)^{1/2} \qquad (c - r < x < c + r)$$

of radius r with center at $(c, 0)$. (We are particularly interested in the cases $r = y_0$, $c = \pm |\mu|$ indicated in Fig. 1.) Such a semi-circle separates the half plane $y \geqslant 0$ into two regions, $\{(x, y): -\infty < x < \infty,\ y \geqslant y_c(x)\}$ and $\{(x, y): c - r < x < c + r, y \leqslant y_c(x)\}$ which we will refer to as "above" and "below" the semi-circle, respectively. The slope of $y_c(x)$ at the point (x, y) is

$$y_c' = \frac{-x + c}{y}.$$

Comparison with (2) shows that if $c > -\mu \sin y$, in particular if $c = |\mu| + \epsilon$ where $\epsilon > 0$ is arbitrarily small, then the trajectory passing through the point (x, y) has slope smaller than y_c', as indicated by the arrows in Fig. 1. It follows that no trajectory can cross a semi-circle with center at $(|\mu| + \epsilon, 0)$ from below to above with increasing time. Similarly no trajectory can cross a semi-circle with center at $(-|\mu| - \epsilon, 0)$ from above to below with increasing time.

The orbit $y(x)$ corresponding to y_0 satisfies $y(|\mu| + \epsilon) < y_0$; a semi-circle $y_{|\mu|+\epsilon}(x)$ of radius y_0 with center at $(|\mu| + \epsilon, 0)$ satisfies $y_{|\mu|+\epsilon}(|\mu| + \epsilon) = y_0$. Therefore $y(x) < y_{|\mu|+\epsilon}(x)$ for $|\mu| + \epsilon \leqslant x$. It follows that the orbit corresponding to y_0 intersects the positive x axis at a point $(b, 0)$ such that $b < y_0 + |\mu| + \epsilon$. See Fig. 1. One proves similarly that the orbit corresponding to y_0 intersects the negative x axis at a point $(a, 0)$ such that $-y_0 - |\mu| - \epsilon < a$. There are simpler ways than the preceding to prove that a and b exist, but we shall have further use for the circles of Fig. 1. We supress the dependence of a and b on y_0 and define

$$\Delta(y_0) = \tfrac{1}{2}(b^2 - a^2). \tag{5}$$

The orbit corresponding to y_0 is periodic if $\Delta(y_0) = 0$, and we shall prove that $\Delta(y_0)$ has an infinite number of zeros. Since Δ is continuous, it will suffice to prove that for large y_0, Δ is alternately positive and negative. Let sgn s denote the sign of s.

THEOREM. *If* $|\mu| < 2$ *then for* $n = 1, 2, \ldots$ *we have*

$$\operatorname{sgn} \Delta(n\pi) = (-1)^n \operatorname{sgn} \mu. \tag{6}$$

Proof. Defining

$$\xi(y) = -x_-(y) - \mu \sin y, \qquad \eta(y) = x_+(y) + \mu \sin y,$$

we have $\xi > 0$, $\eta > 0$, and by (2)

$$g(y) = y/\xi + y/\eta, \quad \xi'(y) = -y/\xi - \mu \cos y, \quad \eta'(y) = -y/\eta + \mu \cos y,$$

so that

$$g'(y) = \frac{1}{\xi^2}\left(\xi + \frac{y^2}{\xi} + \mu y \cos y\right) + \frac{1}{\eta^2}\left(\eta + \frac{y^2}{\eta} - \mu y \cos y\right).$$

Define

$$Q = \frac{\xi^3\eta^3 g'(y)}{\xi + \eta} = y^2(\xi^2 - \xi\eta + \eta^2) + \xi^2\eta^2 + \mu y \xi\eta(\eta - \xi) \cos y.$$

We have

$$Q \geqslant y^2(\xi^2 - \xi\eta + \eta^2) + \xi^2\eta^2 - |\mu| y\xi\eta |\xi - \eta| = P$$

defining P, so that

$$P = \left[y |\xi - \eta| - \frac{|\mu|}{2} \xi\eta\right]^2 + [y^2 + (1 - \mu^2/4)\xi\eta]\xi\eta,$$

which is positive for $|\mu| < 2$. The remainder of the proof we state as a lemma.

LEMMA. *For all μ, if P is positive along the orbit corresponding to $y_0 = n\pi$ then equation* (6) *holds.*

Proof. We have seen above that if P is positive then $g'(y) > 0$, so that g is a strictly increasing function. Using (3) and (5),

$$\begin{aligned} \Delta(n\pi) &= \tfrac{1}{2}(b^2 - a^2) \\ &= u(b, 0) - u(a, 0) \\ &= \int_a^b (-\mu \sin y)\, dx \end{aligned}$$

where the integral is along the orbit corresponding to $y_0 = n\pi$. Thus

$$\begin{aligned} \Delta(n\pi) &= -\mu \left\{\int_0^{n\pi} \sin y \frac{dx_-(y)}{dy} dy + \int_{n\pi}^0 \sin y \frac{dx_+(y)}{dy} dy\right\} \\ &= -\mu \int_0^{n\pi} g(y) \sin y \, dy. \end{aligned} \quad (7)$$

If n is even, say $n = 2k$, then

$$\begin{aligned}\int_0^{2k\pi} g(y)\sin y\,dy &= \sum_{j=0}^{k-1}\int_{2\pi j}^{2\pi(j+1)} g(y)\sin y\,dy\\ &= \sum_{j=0}^{k-1}\left\{\int_{2\pi j}^{\pi(2j+1)} g(y)\sin y\,dy + \int_{\pi(2j+1)}^{2\pi(j+1)} g(y)\sin y\,dy\right\}\\ &= \sum_{j=0}^{k-1}\int_0^{\pi}[g(2\pi j + t) - g(\pi(2j+1)+t)]\sin t\,dt\end{aligned}$$

which is negative since g is increasing. Similarly, if n is odd, say $n = 2k + 1$, then

$$\begin{aligned}\int_0^{(2k+1)\pi} g(y)\sin y\,dy &> \int_{\pi}^{(2k+1)\pi} g(y)\sin y\,dy\\ &= \sum_{j=0}^{k-1}\left\{\int_{\pi(2j+1)}^{2\pi(j+1)} g(y)\sin y\,dy + \int_{2(\pi j+1)}^{\pi(2j+3)} g(y)\sin y\,dy\right\}\\ &= \sum_{j=0}^{k-1}\int_0^{\pi}[-g(\pi(2j+1)+t) + g(2\pi(j+1)+t)]\sin t\,dt\end{aligned}$$

which is positive since g is increasing. Since the sign of the integral in (7) is $(-1)^{n+1}$, the lemma is proved.

Corollary. *If $|\mu| < 2$ then: $\ddot{x} + \mu\sin\dot{x} + x = 0$ has an infinite number of limit cycles. In the phase plane $\dot{x} = y$, $\dot{y} = -\mu\sin y$, there is a limit cycle with initial value $x(0) = x^*$, $y(0) = y^*$ such that $x^* = -\mu\sin y^*$, $n\pi < y^* < (n+1)\pi$ for $n = 1, 2, 3, \ldots$. If $\mu > 0$ then the trajectory with initial value $x(0) = 0$, $y(0) = n\pi$ is an outward spiral for n even and an inward spiral for n odd; these senses of spiraling are reversed if $\mu < 0$.*

Proof. Since (6) shows that $\Delta(y_0)$ is of opposite signs at $n\pi$ and $(n+1)\pi$, and Δ is a continuous function of y_0, Δ must be zero for some y^* such that $n\pi < y^* < (n+1)\pi$. Thus the orbit corresponding to y^* has $|a| = b$. Since the differential equation (2) is symmetric about the origin, the continuation to $y \leqslant 0$ of the orbit corresponding to y^* must return to the negative x-axis at the point $(a, 0)$, thus is a closed curve corresponding to a periodic solution.

Suppose $\mu > 0$ and n even. Then by (6), $\Delta(n\pi)$ is positive, so that for the orbit corresponding to $n\pi$ we have $b > |a|$. By the symmetry of (2), the trajectory in the half plane $y \leqslant 0$ which starts at $(|a|, 0)$ must return to the negative x axis at $(-b, 0)$. The two portions of trajectories mentioned above and the two segments of the x-axis is $[-b, a]$ and $[|a|, b]$ together form a circle homeomorph such that no orbit enters its interior. It follows that the continuation of the orbit corresponding to $n\pi$ into the half plane $y \leqslant 0$ returns to the negative x-axis at a point $(a_1, 0)$ such that $a_1 < -b < a$. Thus for n even, the orbit corresponding to $n\pi$ is an outward spiral. The proof is similar for the other cases mentioned. The corollary is proved.

THEOREM 2. *For arbitrary μ and integers n such that*

$$n > \frac{2}{\pi} | \mu | [| \mu | + 1 + (\mu^2 + 2 | \mu |)^{1/2}],$$

equation (6) holds and, with n so restricted, every statement of the corollary to Theorem 1 holds.

Proof. The proof is the same as that of Theorem 1, using the lemma, except that in the present case it is more work to prove that P is positive. We have, using previous notation,

$$\begin{aligned} P &= y^2(\xi - \eta)^2 + y^2\xi\eta + \xi^2\eta^2 - | \mu | y\xi\eta | \xi - \eta | \\ &\geqslant \xi\eta(y^2 + \xi\eta - | \mu | y | \xi - \eta |) = \xi\eta R \end{aligned}$$

defining R. We shall prove that R is positive.

We begin by returning to the semi-circles of Fig. 1, denoted previously $y_c(x)$. We have seen that the orbit corresponding to y_0 lies below the curve $[x - (| \mu | + \epsilon)]^2 + y^2 = y_0^2$ for each $\epsilon > 0$ and for $| \mu | \leqslant x \leqslant b$. Since ϵ is arbitrary, it follows that

$$x_+(y) \leqslant | \mu | + (y_0^2 - y^2)^{\frac{1}{2}} \tag{8}$$

for $| \mu | \leqslant x_+ \leqslant b$. Since (8) also holds if $| x_+(y)| < | \mu |$, we see that (8) holds for $0 \leqslant y \leqslant y_0$. Similarly we prove for $0 \leqslant y \leqslant y_0$

$$\begin{aligned} -| \mu | - (y_0^2 - y^2)^{\frac{1}{2}} \leqslant x_-(y) \leqslant | \mu | - (y_0^2 - y^2)^{\frac{1}{2}} \\ -| \mu | + (y_0^2 - y^2)^{\frac{1}{2}} \leqslant x_+(y) \leqslant | \mu | + (y_0^2 - y^2)^{\frac{1}{2}} \end{aligned} \tag{9}$$

where x_- and x_+ refer to the orbit corresponding to y_0. Since $x_+(y)$ and $-x_-(y)$ differ by at most $2 | \mu |$ by (9), we have

$$| \xi - \eta | \leqslant 4 | \mu |. \tag{10}$$

We also recall that ξ and η are positive, and see from (9) that both satisfy

$$\xi, \eta \geqslant (y_0^2 - y^2)^{\frac{1}{2}} - 2 | \mu |. \tag{11}$$

We examine two cases. If $(y_0^2 - y^2)^{\frac{1}{2}} - 2 \mid \mu \mid < 0$, then $y^2 > y_0^2 - 4\mu^2$, so that using (9),

$$R > y_0^2 - 4\mu^2 - \mid \mu \mid y_0 4 \mid \mu \mid = y_0^2 - 4\mu^2 y_0 - 4\mu^2,$$

and the latter is a quadratic in y_0 which has a root at

$$y_0 = 2\mu^2 + (4\mu^4 + 4\mu^2)^{\frac{1}{2}} < 2 \mid \mu \mid [\mid \mu \mid + 1 + (\mu^2 + 2 \mid \mu \mid)^{\frac{1}{2}}],$$

and is positive for larger values of y_0, so that case 1 is complete.

If $(y_0^2 - y^2)^{\frac{1}{2}} - 2 \mid \mu \mid \geqslant 0$, then by (10) and (11)

$$\begin{aligned} R &\geqslant y^2 + [(y_0^2 - y^2)^{\frac{1}{2}} - 2 \mid \mu \mid]^2 - 4\mu^2 y \\ &= y_0^2 - 4 \mid \mu \mid (y_0^2 - y^2)^{\frac{1}{2}} + 4\mu^2 - 4\mu^2 y \\ &\geqslant y_0^2 - 4 \mid \mu \mid y_0 + 4\mu^2 - 4\mu^2 y_0 \end{aligned}$$

which is a quadratic in y_0 checked as above to be positive for

$$y_0 > 2 \mid \mu \mid [\mid \mu \mid + 1 + (\mu^2 + 2 \mid \mu \mid)^{\frac{1}{2}}].$$

The proof is complete.

In conclusion, we remark that no attempt has been made to make the constants of Theorems 1 and 2 best possible. It is also easy to improve the bounds (9) by using different circles.

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